

# GUP, Einstein static universe and cosmological constant problem

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We study the Generalized Uncertainty Principle (GUP) in the framework of Einstein static universe (ESU). It is shown that the deformation parameter corresponding to the Snyder non-commutative space can induce an energy density subject to GUP which obeys the holographic principle (HP) and plays the role of a cosmological constant. Using the holographic feature of GUP energy density, we introduce new holographic based IR and UV cut-offs. Moreover, we propose a solution to the cosmological constant problem. This solution is based on the result that the Einstein equations just couple to the tiny holographic based surface energy density (cosmological constant) induced by the deformation parameter, rather than the large quantum gravitational based volume energy density (vacuum energy) having contributions of order  $M_p^4$ .

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## I. INTRODUCTION

In the last decade, a new model of dark energy (DE) so called Holographic Dark Energy (HDE) was proposed based on the holographic principle (HP) [1, 2]. The energy density for HDE is given by [3]

$$\rho_\Lambda = 3c^2 M_p^2 L^{-2}, \quad (1)$$

where  $c$  is a numerical constant,  $M_p$  is the Planck mass and  $L$  is the infrared IR cut-off length. Different choices for IR cut-off length have been proposed such as Hubble length, particle horizon, event horizon, apparent horizon, Ricci radius, Granda-Oliveros cut-off, and so on. On the other hand, the existence of a minimal length is a prediction of quantum theory of gravity [6]-[11]. Therefore, at high energy physics such as early universe we must consider the effects of such a minimal length. Such consideration is achieved by the deformation of standard Heisenberg commutation relation known as the Generalized Uncertainty Principle (GUP) [12, 13]. The simplest form of such relation in the framework of Snyder non-commutative space is given by [14]

$$\Delta q \Delta p \geq \frac{1}{2} | < \sqrt{1 - \alpha p^2} > |, \quad (2)$$

where  $\alpha$  is a deformation parameter which is assumed to be very small. The corresponding commutation relation can be written as

$$[q, p] = i\sqrt{1 - \alpha p^2}, \quad (3)$$

where the only freedom is on the sign of the deformation parameter  $\alpha$ . Therefore, a maximum momentum or a minimal length are predicted by the Snyder-deformed relation (3) if  $\alpha > 0$  or  $\alpha < 0$ , respectively.

On the other hand, the emergent ‘‘Einstein Static Universe’’ (ESU) scenario was proposed by Ellis *et al* to solve the initial singularity problem in the standard cosmological model [15]. In this letter, we study the impact of Generalized Uncertainty Principle (GUP) on the Einstein Static Universe (ESU), Holographic Dark Energy (HDE) and also the Cosmological Constant Problem (CCP), in the context of Emergent Universe.

## II. GUP AND ESU

In this section, we investigate the consistency of the emergent Einstein Static Universe scenario with GUP. Let us consider the isotropic and homogeneous FRW cosmological models described by the line element

$$ds^2 = -N^2 dt^2 + a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (4)$$

where  $N = N(t)$  and  $a = a(t)$  are the lapse function and scale factor, respectively, and the spatial curvature  $k$  can be zero or  $\pm 1$  depending on the symmetry group. The dynamics of these models is described by the constraint

$$\mathcal{H} = -\frac{2\pi G}{3} \frac{p_a^2}{a} - \frac{3}{8\pi G} a k + a^3 \rho = 0, \quad (5)$$

where  $G$  is the gravitational constant,  $\rho = \rho(a)$  is the energy density of a typical perfect fluid, and  $p_a$  is the momentum conjugate of the scale factor  $a$ . The phase space of this system has 2-dimensions in which the only non-vanishing Poisson bracket is  $\{a, p_a\} = 1$ . Using the Hamilton equations, the Friedmann equation can be obtained as follows

$$\mathcal{H}_E = \frac{2\pi G}{3} N \frac{p_a^2}{a} + \frac{3}{8\pi G} N a k - N a^3 \rho + \lambda \pi, \quad (6)$$

where  $\mathcal{H}_E$  is the extended Hamiltonian,  $\lambda$  is a Lagrange multiplier and  $\pi$  is the momenta conjugate to the lapse

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function. The equations of motion with respect to  $\mathcal{H}_E$  are obtained as

$$\begin{aligned}\dot{N} &= \{N, \mathcal{H}_E\} = \lambda, \\ \dot{a} &= \{a, \mathcal{H}_E\} = \frac{4\pi G}{3} N \frac{p_a}{a}, \\ \dot{p}_a &= \{p_a, \mathcal{H}_E\} \\ &= N \left( \frac{2\pi G}{3} \frac{p_a^2}{a^2} - \frac{3}{8\pi G} k + 3a^2 \rho + a^3 \frac{d\rho}{da} \right).\end{aligned}\quad (7)$$

By using the above equations and the constraint (5), we can obtain the Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}.\quad (8)$$

Now, we study the analysis of deformed dynamics of the FRW model to obtain the modifications of the Friedmann equation resulting from the algebra (3). The modified symplectic geometry at the classical limit of the algebra (3), results in the Snyder-deformed classical dynamics. According to Dirac, we can replace the quantum-mechanical commutator (3) by the classical Poisson bracket

$$-i[\tilde{q}, p] \implies \{\tilde{q}, p\} = \sqrt{1 - \alpha p^2}.\quad (9)$$

The deformed Poisson bracket must satisfy the same properties as those of quantum mechanical commutator, i.e. it has to be bilinear and anti-symmetric, and also it must satisfy the Leibniz rules as well as the Jacobi identity. Thus, the deformed Poisson bracket in two-dimensional phase space is given by

$$\{F, G\} = \left( \frac{\partial F}{\partial \tilde{q}} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial \tilde{q}} \right) \sqrt{1 - \alpha p^2}.\quad (10)$$

Specially, the canonical equations for coordinate and momentum from the deformed Hamiltonian  $\mathcal{H}(\tilde{q}, p)$  are obtained

$$\begin{aligned}\dot{\tilde{q}} &= \{\tilde{q}, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p} \sqrt{1 - \alpha p^2}, \\ \dot{p} &= \{p, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial \tilde{q}} \sqrt{1 - \alpha p^2}.\end{aligned}\quad (11)$$

Now, we apply this deformation scheme to the FRW model in the presence of matter energy density, namely to the Hamiltonian (6). We assume the minisuperspace to be Snyder-deformed and consequently the commutator between the scale factor  $a$  and its conjugate momentum  $p_a$  is uniquely obtained as

$$\{a, p_a\} = \sqrt{1 - \alpha p_a^2},\quad (12)$$

which does not change the equations of motion  $\dot{N} = \{N, \mathcal{H}_E\} = \lambda$  and  $\dot{\pi} = \{\pi, \mathcal{H}_E\} = \mathcal{H} = 0$ . The Poisson bracket  $\{N, \pi\} = 1$  is not affected by the deformations induced by the  $\alpha$  parameter. Nevertheless, the equations

of motion (7) are modified in this approach via the relation (12), so we have

$$\begin{aligned}\dot{a} &= \{a, \mathcal{H}_E\} = \frac{4\pi G}{3} N \frac{p_a}{a} \sqrt{1 - \alpha p_a^2}, \\ \dot{p}_a &= \{p_a, \mathcal{H}_E\} = \sqrt{1 - \alpha p_a^2} \times \\ &\quad N \left( \frac{2\pi G}{3} \frac{p_a^2}{a^2} - \frac{3}{8\pi G} k + 3a^2 \rho + a^3 \frac{d\rho}{da} \right).\end{aligned}\quad (13)$$

The deformed Friedmann equation can be obtained by solving the constraint (5) with respect to  $p_a$  and considering the first equation of (13) as follows ( $N = 1$ )

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{8\pi G}{3} \rho - \frac{k}{a^2}\right) \left[1 - \frac{3\alpha}{2\pi G} a^2 \left(a^2 \rho - \frac{3}{8\pi G} k\right)\right].\quad (14)$$

The conservation equations for the matter part is also given by

$$\dot{\rho} + 3H(\rho + p) = 0.\quad (15)$$

For the flat FRW universe ( $k = 0$ ), the equation (14) can be written as [22]

$$\left(\frac{\dot{a}}{a}\right)_{k=0}^2 = \frac{8\pi G}{3} \rho \left(1 - \text{sgn } \alpha \frac{\rho}{\rho_c}\right),\quad (16)$$

where  $\rho_c = (2\pi G/3|\alpha|)\rho_P$  is the critical energy density and  $\rho_P$  is the Planck energy density, where in the last step we have assumed the existence of a fundamental minimal cut-off length which should be related to the Planck cut-off (for a review see [45]). The modifications in the deformed Friedmann equation (16) are manifested in the form of a  $\rho^2$ -term. This factor is important at high energy regime such that if  $\alpha > 0$  and  $\rho$  reaches the critical value  $\rho_c$ , the Hubble parameter ( $\dot{a}/a$ ) vanishes and the universe experiences a bounce (or more generally a turn-around) in the scale factor. For  $\rho \ll \rho_c$ , we recover the standard Friedmann dynamics, namely equation (8) for  $k = 0$ . In the same way, when  $\alpha$  vanishes, the correction term disappears and the ordinary behavior of the Hubble parameter is recovered.

By using the equations (14) and (15) for the closed FRW universe ( $k = 1$ ) and the equation of state  $p = w\rho$ , the acceleration equation is obtained <sup>1</sup>

$$\begin{aligned}\ddot{a} &= \left[ -\frac{a}{6} + \frac{27}{4} \alpha a^3 - \frac{3a}{2} w \left( \frac{1}{3} + \frac{27}{2} \alpha a^2 \right) \right] \rho \\ &\quad + 12\alpha a^5 w \rho^2 - \frac{9\alpha a}{2},\end{aligned}\quad (17)$$

where the matter energy density  $\rho$  is given by solving the equation (14) as follows

$$\begin{aligned}\rho &= \frac{1}{48a^6\alpha} (81a^4\alpha + 2a^2 \\ &\quad \pm \sqrt{-576a^8H^2\alpha + 3969a^8\alpha^2 - 252a^6\alpha + 4a^4}).\end{aligned}\quad (18)$$

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<sup>1</sup> We have used the units  $8\pi G = M_P^{-1} = 1$ .

Imposing the requirements of Einstein static solution given by  $\ddot{a} = \dot{a} = 0$  in the equations (17) and using (18), we obtain the solutions  $a(\alpha)$  and  $a(\alpha, \omega)$  and the corresponding energy densities  $\rho_\alpha$  and  $\rho_\alpha(\omega)$ , respectively as follows

$$a_\alpha^2 = \frac{2}{63\alpha}, \quad \rho_\alpha = \frac{189}{2}\alpha, \quad (19)$$

and

$$a_\alpha^2(\omega) = \frac{2(1-3w)}{9\alpha(1+3w)} \quad \rho_\alpha(\omega) = \frac{27\alpha(1+3w)}{8(1-3w)^2}, \quad (20)$$

where we call  $\rho_\alpha$  and  $\rho_\alpha(\omega)$  as GUP energy densities. For a positive  $\alpha$ , the reality condition for  $a_\alpha$  and the positivity condition of  $\rho_\alpha$  result in the following domain for the equation of state parameter

$$-1/3 < w < 1/3. \quad (21)$$

By removing the parameter  $\alpha$ , the equations (19) and (20) can be rewritten respectively as

$$\rho_\alpha = \frac{3}{a_\alpha^2}, \quad (22)$$

and

$$\rho_\alpha(\omega) = \frac{\rho_\alpha}{4(1-3w)} = \frac{3}{4a_\alpha^2(1-3w)}. \quad (23)$$

Considering the fact that the parameter  $\alpha$  is very small, we conclude that the energy density  $\rho_\alpha$  in (19) is very small, and the size of corresponding Einstein static universes  $a_\alpha$  is very large. Actually, this result is not consistent with the paradigm of Einstein static universe in the context of emergent universe scenario because this scenario is supposed to be implemented at early universe having very small size to solve the initial singularity problem, whereas the solution (19) corresponds to the Einstein static universe having a very large size. In other words, this solution for Einstein static universe as an emergent universe is inconsistent with the implementation of GUP. Therefore, we ignore this solution as a nonviable solution in the context of emergent universe. However, the size of Einstein static universe  $a_\alpha(\omega)$  corresponding to  $\rho_\alpha(\omega)$  can be sufficiently small, by appropriate choice of  $\omega \lesssim \frac{1}{3}$ , such that it can be considered as an emergent Einstein static universe at very early time.

### III. GUP, ESU AND IR-UV CUT-OFFS FOR HDE

Comparing (22) and (23) with the generic form of the HDE, described by the equation (1), it is obvious that the energy densities corresponding to the Einstein static universe show a holographic feature. In the previous section, we showed that the radius of Einstein static universe  $a_\alpha$

is so large that cannot be considered as an emergent universe. So, one may ask about the physical importance of  $a_\alpha$ . In this regard, assuming  $c^2 = 1, M_p^{-1} = 1$ , the most natural interpretation for  $a_\alpha$  in the context of holographic dark energy is that the radius  $a_\alpha$  may be considered as a new cut-off length  $L = a_{Es}$  which is an IR cut-off. Considering the other solution (23), one may also assume that  $L = 2a_{Es}\sqrt{(1-3w)}$  is an IR cut-off for most of allowed values of  $\omega$  within the interval  $-1/3 < w < 1/3$ , with the exception that it can also yield an  $w$  dependent UV cut-off for  $\omega \lesssim \frac{1}{3}$ .

### IV. GUP, ESU AND CCP

The equations of motion (14) and (17) in the absence of  $\alpha$  and without a cosmological constant  $\Lambda$  lead to the usual Friedmann equations which describe an expanding universe. However, in the presence of  $\alpha$  and without a cosmological constant  $\Lambda$ , we obtain Einstein static universe. This clearly indicates that in the absence of cosmological constant  $\Lambda$ , the static state of Einstein universe can be provided by the quantum gravitational effect induced by the deformation parameter  $\alpha$ . In other words, the deformation parameter  $\alpha$  induces energy densities  $\rho_\alpha$  and  $\rho_\alpha(\omega)$  which can play the role of cosmological constants capable of constructing the Einstein static universe. These energy densities obey the holographic principle and ensures us about the fact that the origin of these energy densities lie on the two dimensional surface of a sphere with the radius of Einstein static universe. The bounds on  $\alpha$  should be a subject of precise cosmological observations, for instance the measurement of the cosmological constant.

For the case of  $\rho_\alpha$ , by identification of  $\rho_\alpha$  with the cosmological constant we can estimate the value of deformation parameter  $\alpha$  by considering the observational upper bound on the present value of cosmological constant as  $\alpha \sim (\Lambda \sim 10^{-47} \text{GeV}^4)$ . This also gives estimation on the size of the radius of Einstein static universe as  $a_\alpha = a_E \sim 10^{28} m$  which coincides with the size of the observable universe. We may also study the cosmological constant problem in the present model for the energy density  $\rho_\alpha$ . The cosmological constant problem arises because of the huge disagreement between the observed value of cosmological constant, of order  $10^{-47} \text{GeV}^4$ , and the zero-point energy suggested by quantum field theory (QFT). In fact, a large discrepancy of order  $10^{121}$  is shown up when we consider the quantum gravitational corrections to the zero-point energy of order  $M_p^4 \sim 10^{74} \text{GeV}^4$ . The zero-point energy in quantum field theory is provided by the vacuum quantum field fluctuations originated by the Heisenberg uncertainty principle  $\Delta q \Delta p \geq \frac{1}{2}$ . By applying GUP as  $\Delta q \Delta p \geq \frac{1}{2} |\langle \sqrt{1 - \alpha p^2} \rangle|$ , we expect that a correction term caused by deformation parameter  $\alpha$  should contribute a correction term to the zero-point energy. We may estimate the order of magnitude of the correction

term as  $\alpha p^2 \sim \alpha M_P^2 \sim 10^{-10} \ll 1$ . Therefore, it turns out that the contribution of this correction term to the zero-point energy of quantum gravity is negligible, such that GUP cannot affect the order of magnitude of quantum gravitational zero-point energy  $M_P^4 \sim 10^{74} \text{GeV}^4$ . In other words, the cosmological constant problem still exists in the framework of GUP. However, the interesting point is that the scale of energy density  $\rho_\alpha$  appearing as the cosmological constant term in the Einstein (Friedmann) equations, corresponding to the deformation parameter  $\alpha$ , is of order  $10^{-47} \text{GeV}^4$ . This can be considered as a solution of the cosmological constant problem in the present Einstein static universe model, as is described in the following. We know that the zero-point energy density is the zero-point energy per unit volume, namely it is a energy density distributed over the entire volume inside the Einstein static universe, whereas the energy density  $\rho_\alpha$  corresponding to the deformation parameter  $\alpha$  is a surface energy density distributed over the entire surface enclosing the Einstein static universe. Since the cosmological constant in the Einstein equation (which arises effectively from GUP in the Einstein static universe) is naturally set by the holographic surface energy density  $\rho_\alpha$  distributed over the surface enclosing the Einstein static universe, it is reasonable to assume that the holographic based cosmological constant  $\Lambda \sim \rho_\alpha$  with the origin over the enclosing surface of Einstein static universe cannot be affected by the zero-point volume energy density raised by the quantum gravitational corrections of order  $M_P^4$  distributed over the volume of Einstein static universe. In simple words, it seems that the observed cosmological constant  $\Lambda \sim \rho_\alpha$  has just a holographic nature which is merely provided by the information encoded on the surface of the sphere enclosing the Einstein static universe, rather than the volume information inside the Einstein static universe. Therefore, one may conclude that the volume information including the quantum gravitational perturbative corrections, in principle cannot contribute to the surface information including the classical energy density  $\rho_\alpha$  appearing in the Einstein equations. In other words, it turns out that the Einstein tensor just feels (and react against) the small holographic surface energy density of order  $10^{-47} \text{GeV}^4$  rather than the huge volume energy density of order  $10^{74} \text{GeV}^4$ . Therefore, similar to the quantum field theory in flat spacetime where the zero of energy is arbitrary and the vacuum energy with infinite energy is considered as a nonphysical background energy, the volume energy density here which is not felt by the Einstein tensor, can also be regarded as a nonphysical background energy which is easily discarded by some mathematical techniques (for instance, normal ordering). It seems that the cosmological constant problem can be solved somehow in a way similar to the case that one extracts for example the tiny physical mass or charge of an electron from the large bare quantities via renormalization procedure. What happens in this model is that the Einstein equations naturally couples to the tiny holographic based physical cosmological constant  $\Lambda \sim \rho_\alpha$  ex-

tracted from the bare volume energy density having large contributions from quantum gravitational corrections.

It is well-known that the zero-point energy is fundamentally related to the Heisenberg Uncertainty Principle stated as  $\Delta q \Delta p \geq \frac{1}{2}$  or the commutation relation  $[q, p] = i$ . Since the zero-point energy is distributed over the entire space, it is obviously a volume energy density which can be attributed to “ $\frac{1}{2}$ ” or “ $i$ ” in  $\Delta q \Delta p \geq \frac{1}{2}$  or  $[q, p] = i$ , respectively. Considering the Generalized Uncertainty Principle  $\Delta q \Delta p \geq \frac{1}{2} |\langle \sqrt{1 - \alpha p^2} \rangle|$  or generalized commutation relation  $[q, p] = i \sqrt{1 - \alpha p^2}$ , one can still consider the zero-point energy as a volume energy attributed to “1” appearing within the square root term in  $\frac{1}{2} |\langle \sqrt{1 - \alpha p^2} \rangle|$  or  $i \sqrt{1 - \alpha p^2}$ . However, we have already shown that the second term within the square root terms, having contributions of deformation parameter  $\alpha$ , corresponds to a holographic surface energy density. Therefore, one may interpret the terms “1” and “ $\alpha p^2$ ” corresponding to the bare volume energy density and physical surface energy density, respectively. The solution of cosmological constant problem lies in the fact that the bare cosmological constant of order  $M_P^4$  coming from the volume energy density associated to “1” in the Generalized Uncertainty Principle does not contribute to the holographic based physical cosmological constant  $\Lambda \sim \rho_\alpha$  coming from the deformation parameter  $\alpha$ . In fact, the physical cosmological constant which is coupled to the Einstein equations comes from the holographic surface energy density, and the bare cosmological constant having huge contributions from quantum gravitational corrections is naturally ruled out from being coupled to the Einstein equations.

For the case of  $\rho_\alpha(\omega)$ , the holographic energy density depends on both the quantum deformation parameter  $\alpha$  and the classical equation of state parameter  $\omega$ . For any value of  $\omega$  given by (21), the radius of Einstein static universe and the holographic energy density are determined by the equation (20). Moreover, according to the equation (23), the variable holographic energy density  $\rho_\alpha(\omega)$  is proportional to the constant holographic energy density  $\rho_\alpha$ . Therefore, one may write  $\rho_\alpha(\omega) = \rho_\alpha / 4(1 - 3\omega) = \rho_\alpha^{eff}$  where  $\rho_\alpha^{eff}$  is considered as *effective* cosmological constant. Thus, the same discussion on the holographic origin of the observed cosmological constant and the solution of cosmological constant problem can be applied to this case, as well.

## Summary and Conclusion

We have studied the Generalized Uncertainty Principle in the framework of Einstein static universe. We have shown that the deformation parameter can induce an energy density subject to GUP which obeys the holographic feature and plays the role of a cosmological constant. Using the holographic feature of GUP energy density, we have introduced new holographic based IR and UV cutoffs. Moreover, we have realized that the Einstein equa-



tions are naturally coupled to the tiny holographic surface energy density (physical quantity) instead of large volume energy density (bare quantity). Then, we have proposed a solution to the cosmological constant problem which is similar in essence to the case that one extracts the tiny physical quantities from large bare quantities. The emergence of tiny holographic surface energy density (physical cosmological constant) from large volume energy density (bare vacuum energy) seems to be a kind of natural renormalization through the implementation

of GUP on the Einstein Static Universe.

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